Homework 6

3.3 <u>2d and 3d</u> $\begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{pmatrix} \to$

$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $x_1 = 0$, $x_2 - x_3 = 0$

general solution $\begin{pmatrix} 0 \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\therefore \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ basis of solution space

<u>7a</u>

$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 2 \end{pmatrix}$ compute $r = \text{rank } A =$ dim of row space

$R_3 = R_1 + R_2$ $\quad R_1$ and R_2 independent

$\therefore$ dim row space $= 2$ $\quad r = 2$

Now compute rank $(A|b) = \begin{pmatrix} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{pmatrix}$ R_1, R_2 linearly independent

Is $R_3 = x R_1 + y R_2$ for some x, y? Suppose so

$x + y = 2$, $-x + 2y = 1$ $\quad \therefore x = 1, y = 1$

Also $2x + y = 4$ ~~can~~ incompatible with $x = y = 1$. $\therefore$ No such x, y exists $\therefore R_1, R_2, R_3$ linearly independent

so rank $(A|b) = 3$ $\quad \therefore$ no solution.

<u>8b</u> Matrix of T

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix}$ $\quad (A|v) = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

row reduce A $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ rank 2

$$(A|v) \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ rank } 2$$

$\operatorname{rank} A = \operatorname{rank}(A|v)$ $\therefore$ solution exists

9. $Ax = b$ has solution $\Leftrightarrow \exists s,\ L_A(s) = As = b \Leftrightarrow$ $b \in R(L_A)$.

10 A $m \times n$ matrix, rank m $L_A : F^n \to F^m$

rank $L_A = m$ $\therefore L_A$ is onto $\therefore$ a solution exists by problem 9.

3.4 2b

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & -3 & 1 & 6 \\ 3 & -5 & 0 & 7 \\ 1 & 0 & 5 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 6 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 5 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 &= 9 - x_3 \\ x_2 &= 4 - 3x_3 \end{aligned} \quad \text{for any } x_3$$

2f

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 3 & 3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} x_1 &= -3 + x_4 \\ x_2 &= 3 - 2x_4 \\ x_3 &= 1 \end{aligned}$$

7 $\{u_1, \ldots, u_5\}$ is not a basis so there exists $c_1, \ldots, c_5$

$$c_1(2,-3,1) + c_2(1,4,-2) + c_3(-8,12,-4) + c_4(1,37,-17) +$$

$$c_5(-3,-5,8) = 0$$

This gives a homogeneous system with unknown $c_1, \ldots, c_5$ whose augmented matrix is

$$A = \begin{pmatrix} 2 & 1 & -8 & 1 & -3 & 0 \\ -3 & 4 & 12 & 37 & -5 & 0 \\ 1 & -2 & -4 & -17 & 8 & 0 \end{pmatrix}$$ Put A in RRE form

$$A \to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 & 0 \\ 0 & 2 & 0 & 14 & -19 & 0 \\ 0 & 5 & 0 & 35 & -19 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & -3 & -11 & 0 \\ 0 & 1 & 0 & 7 & -\frac{19}{2} & 0 \\ 0 & 5 & 0 & 35 & -19 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -4 & -3 & 0 & 0 \\ 0 & 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$ Columns 1, 2 and 5 of this last matrix are linearly independent

$\therefore \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 8 \end{pmatrix}$ are linearly independent

4.4 4e $i \begin{vmatrix} 1+i & 2 \\ 1 & 4-i \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ -2i & 4-i \end{vmatrix} - \begin{vmatrix} 3 & 1+i \\ -2i & 1 \end{vmatrix}$

$= i\left((i+1)(-i+4) - 2\right) - 2(12 - 3i + 4i) - (3 + 2i(i+1))$

$= -(i + 28)$.

5 A is $r \times r$, $I = I_s$ $n = r+s$ B is $r \times s$ O is $s \times r$

Proof by induction ~~of~~ on s $s=1$ clear. Assume true for $s-1$ and expand $\det M$ along bottom row getting

$\det M = (-1)^{2n} \det \begin{pmatrix} A & B' \\ 0 & I_{s-1} \end{pmatrix}$ where B' is B with last column deleted.

$\therefore B'$ is $r \times s-1$ ~~xxx~~ By induction,

$= \det \begin{pmatrix} A & B' \\ 0 & I_{s-1} \end{pmatrix} = \det A.$